

RAREFIED ELLIPTIC HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. We prove exact evaluation formulae for two multiple rarefied elliptic beta integrals related to the simplest lens space. These integrals generalize the multiple type I and II van Diejen-Spiridonov integrals attached to the root system C_n . Symmetries of the rarefied elliptic analogue of the Euler-Gauss hypergeometric function are described and the corresponding generalization of the hypergeometric equation is constructed. An extension of the latter function to the root system C_n and applications to some eigenvalue problems are briefly discussed.

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1. INTRODUCTION

Hypergeometric functions lie at the center of the world of special functions [1]. Elliptic functions (i.e., the meromorphic doubly periodic functions) form another key representative from this world. These two classical sets of functions appeared to be deeply tied to each other, which became clear after the discovery of elliptic hypergeometric functions [17]. Elliptic hypergeometric integrals [12] describe currently the most general known functions of hypergeometric type which are transcendental over the fields of elliptic, ordinary hypergeometric and q -hypergeometric functions. Unification of elliptic and hypergeometric wisdoms elucidated various previously known properties of the corresponding functions. The unique nature of the most interesting elliptic hypergeometric functions is established by their symmetries associated with two independent elliptic curves and two independent root systems [14, 17].

The elliptic beta integral [12] is until now the only known computable integral among univariate elliptic hypergeometric integrals. Its evaluation formula (see (12)) can be interpreted as an elliptic analogue of the Newton's binomial theorem, top extension of the Euler's beta function, normalization of the biorthogonality measure for particular elliptic functions (actually, the product of two elliptic functions with different nomes [13]) forming the most general set of special functions, which extend the Jacobi and Askey-Wilson polynomials [1], etc. It has found remarkable applications in theoretical physics, e.g., in quantum mechanical eigenvalue problems [14, 16]. The most important physical interpretation of the formula (12) was found in quantum field theory, where it proves the equality of superconformal indices of two nontrivial four dimensional supersymmetric models connected by the Seiberg duality and gives the most rigorous mathematical confirmation of the confinement phenomenon (see [10] for a recent survey of this subject).

The next important representative of univariate elliptic hypergeometric integrals is an elliptic analogue of the Euler-Gauss hypergeometric function. It contains more free parameters than the elliptic beta integral, has symmetry transformations related to the exceptional root system E_7 , and satisfies an elliptic hypergeometric equation [13, 14]. Another direction of complication of these functions calls for a consideration of multiple integrals. Here, the very first multiple elliptic beta integrals were defined by van Diejen and the author [6, 7] in relation to the root system C_n . The integrals defined in [7] are referred to as of type I, and those of [6] as of type II (the latter integrals are generalizations of the Selberg integrals). This classification is inspired by differences in the methods used for proving corresponding exact evaluation formulae [7, 9, 15]. Nowadays, the number of known different elliptic hypergeometric integrals admitting (proven or conjectural) either exact evaluation or obeying a nontrivial symmetry transformation described by the Weyl groups of various root systems is very large, see [9, 19, 20].

The standard elliptic hypergeometric integrals are related to the manifold $S^1 \times S^3$, which plays a role of the compact space-time for four dimensional superconformal field theories [10]. However, this is only one of many admissible four dimensional manifolds for which one can compute superconformal indices. The next level of topological complication is related to the replacement of the S^3 -factor by the lens spaces, and the first step along this line has been done in [3], where an analogue of the elliptic gamma function for the simplest lens space has been introduced. Some further essential developments of this subject can be found in [11]. Recently, Kels [8] proposed an extension of the univariate elliptic beta integral associated with the simplest lens space. It involves some additional discrete variables and a replacement of the single integration by a finite sum of integrations. In this work we reconsider the problem of finding an analogue of formula (12) related to the considerations of [3] and find the result which is different from the one in [8]. We show that the general class of such integrals, which we call "rarefied elliptic hypergeometric functions", matches with the general definition of elliptic hypergeometric functions [13] applied to the case when one works with a sum of integrals. From the principal point of view, these functions should be considered as two-variate "sum-integral" objects. Further, we propose two explicit multiple integrals of such type as a generalization of the type I and

II elliptic beta integrals of van Diejen and the author [6, 7]. We consider also a rarefied elliptic analogue of the Euler-Gauss hypergeometric function and derive the corresponding elliptic hypergeometric equation. Finally, an extension of the latter function to the root system C_n is constructed, an analogue of the Rains symmetry transformation [9] for it is conjectured, and its application to the eigenvalue problem for a finite-difference operator is briefly discussed. In the concluding section we outline some prospects for further development of the derived results.

2. THE ELLIPTIC BETA INTEGRAL

In this section we describe the elliptic hypergeometric integrals of [6, 7, 12]. For $p \in \mathbb{C}$, $|p| < 1$, we introduce the standard infinite product

$$(z; p)_\infty = \prod_{j=0}^{\infty} (1 - zp^j), \quad z \in \mathbb{C}. \quad (1)$$

The theta function

$$\theta(z; p) = (z; p)_\infty (pz^{-1}; p)_\infty, \quad z \in \mathbb{C}^*, \quad (2)$$

obeys the following symmetry properties

$$\theta(x^{-1}; p) = \theta(px; p) = -x^{-1}\theta(x; p).$$

We shall need the general relation

$$\theta(p^k z; p) = (-z)^{-k} p^{-\frac{k(k-1)}{2}} \theta(z; p), \quad k \in \mathbb{Z}.$$

The “addition law” for theta functions has the form

$$\theta(xw^{\pm 1}, yz^{\pm 1}; p) - \theta(xz^{\pm 1}, yw^{\pm 1}; p) = yw^{-1}\theta(xy^{\pm 1}, wz^{\pm 1}; p),$$

where $x, y, w, z \in \mathbb{C}^*$. We use the convention

$$\theta(x_1, \dots, x_k; p) = \prod_{j=1}^k \theta(x_j; p), \quad \theta(tx^{\pm 1}; p) = \theta(tx, tx^{-1}; p).$$

For arbitrary $q \in \mathbb{C}$ and $n \in \mathbb{Z}$, the elliptic Pochhammer symbol is defined as

$$\theta(x; p|q)_n := \begin{cases} \prod_{j=0}^{n-1} \theta(xq^j; p), & \text{for } n > 0 \\ \prod_{j=1}^{-n} \theta(xq^{-j}; p)^{-1}, & \text{for } n < 0 \end{cases}$$

and $\theta(x; p|q)_0 = 1$.

The first order q -difference equation

$$f(qz) = \theta(z; p)f(z), \quad q \in \mathbb{C}, \quad (3)$$

has a particular solution

$$\Gamma(z; p, q) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1}p^{j+1}q^{k+1}}{1 - zp^jq^k}, \quad |p|, |q| < 1, \quad z \in \mathbb{C}^*, \quad (4)$$

called the elliptic gamma function. It has the following properties

$$\Gamma(z; p, q) = \Gamma(z; q, p), \quad (5)$$

$$\Gamma(qz; p, q) = \theta(z; p)\Gamma(z; p, q), \quad \Gamma(pz; p, q) = \theta(z; q)\Gamma(z; p, q), \quad (6)$$

$$\Gamma(z; p, q)\Gamma\left(\frac{pq}{z}; p, q\right) = 1, \quad (7)$$

$$\Gamma(\sqrt{pq}; p, q) = 1, \quad (8)$$

$$\lim_{z \rightarrow 1} (1 - z)\Gamma(z; p, q) = \frac{1}{(p; p)_\infty (q; q)_\infty}. \quad (9)$$

The poles and zeros of this function form the double base geometric progressions

$$z_{poles} = p^{-j}q^{-k}, \quad z_{zeros} = p^{j+1}q^{k+1}, \quad j, k \in \mathbb{Z}_{\geq 0}. \quad (10)$$

The elliptic Pochhammer symbol can be written in the form

$$\theta(z; p|q)_m = \frac{\Gamma(zq^m; p, q)}{\Gamma(z; p, q)}, \quad m \in \mathbb{Z}. \quad (11)$$

Define also the elliptic gamma function of the second order

$$\Gamma(z; p, q, t) = \prod_{j,k,l=0}^{\infty} (1 - zp^j q^k t^l)(1 - z^{-1}p^{j+1}q^{k+1}t^{l+1}), \quad |t|, |p|, |q| < 1,$$

satisfying the equations

$$\frac{\Gamma(qz; p, q, t)}{\Gamma(z; p, q, t)} = \Gamma(z; p, t), \quad \frac{\Gamma(pz; p, q, t)}{\Gamma(z; p, q, t)} = \Gamma(z; q, t), \quad \frac{\Gamma(tz; p, q, t)}{\Gamma(z; p, q, t)} = \Gamma(z; p, q)$$

and the inversion relation

$$\Gamma(pqtz; p, q, t) = \Gamma(z^{-1}; p, q, t).$$

The elliptic beta integral evaluation formula, which serves as a germ for the whole general theory of elliptic hypergeometric integrals, has the following form [12].

Theorem 1. *We take eight parameters $t_1, \dots, t_6, p, q \in \mathbb{C}$ with $|t_a|, |p|, |q| < 1$ and $\prod_{a=1}^6 t_a = pq$. Then*

$$\frac{(p; p)_\infty (q; q)_\infty}{2} \int_{\mathbb{T}} \frac{\prod_{a=1}^6 \Gamma(t_a z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{2\pi i z} = \prod_{1 \leq a < b \leq 6} \Gamma(t_a t_b; p, q), \quad (12)$$

where \mathbb{T} is the unit circle of positive orientation.

Here we use the compact notation

$$\begin{aligned} \Gamma(t_1, \dots, t_n; p, q) &:= \Gamma(t_1; p, q) \dots \Gamma(t_n; p, q), \quad \Gamma(tz^{\pm k}; p, q) := \Gamma(tz^k; p, q)\Gamma(tz^{-k}; p, q), \\ \Gamma(z^{\pm 1}w^{\pm 1}; p, q) &:= \Gamma(zw; p, q)\Gamma(z^{-1}w; p, q)\Gamma(zw^{-1}; p, q)\Gamma(z^{-1}w^{-1}; p, q). \end{aligned}$$

In [6, 7], van Diejen and the author proposed two multiple generalizations of the integration formula (12) in association with the root system C_n . The type I integral has the

following form. Let $z_1, \dots, z_n \in \mathbb{T}$ and complex parameters t_1, \dots, t_{2n+4} and p, q satisfy the constraints $|p|, |q|, |t_a| < 1$ and $\prod_{j=1}^{2n+4} t_a = pq$. Then

$$\begin{aligned} \kappa_n \int_{\mathbb{T}^n} \prod_{1 \leq j < k \leq n} \frac{1}{\Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)} \prod_{j=1}^n \frac{\prod_{a=1}^{2n+4} \Gamma(t_a z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^n \frac{dz_j}{z_j} \\ = \prod_{1 \leq a < b \leq 2n+4} \Gamma(t_a t_b; p, q), \quad \kappa_n = \frac{(p; p)_\infty (q; q)_\infty}{(4\pi i)^n n!}. \end{aligned} \quad (13)$$

The type II van Diejen-Spiridonov integral has a structurally different form. Let complex parameters $t, t_a (a = 1, \dots, 6), p$ and q satisfy conditions $|p|, |q|, |t|, |t_a| < 1$, and $t^{2n-2} \prod_{a=1}^6 t_a = pq$. Then

$$\begin{aligned} \kappa_n \int_{\mathbb{T}^n} \prod_{1 \leq j < k \leq n} \frac{\Gamma(t z_j^{\pm 1} z_k^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 1} z_k^{\pm 1}; p, q)} \prod_{j=1}^n \frac{\prod_{a=1}^6 \Gamma(t_a z_j^{\pm 1}; p, q)}{\Gamma(z_j^{\pm 2}; p, q)} \prod_{j=1}^n \frac{dz_j}{z_j} \\ = \prod_{j=1}^n \left(\frac{\Gamma(t^j; p, q)}{\Gamma(t; p, q)} \prod_{1 \leq a < b \leq 6} \Gamma(t^{j-1} t_a t_b; p, q) \right). \end{aligned} \quad (14)$$

We note that the latter integral can be interpreted as an elliptic extension of the Selberg integral [1]. For $n = 1$ both these multiple integrals reduce to the elliptic beta integral (12).

3. THE RAREFIED ELLIPTIC GAMMA FUNCTION

Recently, in [3] an analogue of the elliptic gamma function for the simplest lens space was introduced. Some integrals involving this function were considered in [8, 11]. In comparison to the standard elliptic hypergeometric integrals, they contain additional integer parameters and involve finite summations over discrete variables additional to the standard integrations. In this work we use the analysis of [3, 8, 11] as an inspiration for considering general structure of elliptic hypergeometric integrals of such type, which we call the *rarefied* elliptic hypergeometric integrals.

A lens space elliptic gamma function introduced in [3] is given by a particular product of two standard elliptic gamma functions with different bases

$$\begin{aligned} \gamma^{(r)}(z, m; p, q) &:= \Gamma(z p^m; p^r, pq) \Gamma(z q^{r-m}; q^r, pq) \\ &= \prod_{j,k=0}^{\infty} \frac{1 - z^{-1} p^{-m} (pq)^{j+1} p^{r(k+1)}}{1 - z p^m (pq)^j p^{rk}} \frac{1 - z^{-1} q^m (pq)^{j+1} q^{rk}}{1 - z q^{r-m} (pq)^j q^{rk}}, \end{aligned} \quad (15)$$

which, in addition to the variable $z \in \mathbb{C}^*$, involves two integers $r \in \mathbb{Z}_{>0}$ and $m \in \mathbb{Z}$. It has poles at the points

$$z_{poles} = p^{-m-j-rk} q^{-j}, \quad p^{-j} q^{m-r-j-rk}, \quad (16)$$

and zeros

$$z_{zeros} = p^{j+1+r(k+1)-m} q^{j+1}, \quad p^{j+1} q^{m+j+1+rk}, \quad j, k \in \mathbb{Z}_{\geq 0}. \quad (17)$$

According to [3], this function is associated with the superconformal index of a chiral superfield on the space-time $S^1 \times L(r, 1)$, where $L(r, 1)$ is the lens space defined by the identification of points $(e^{2\pi i/r} z_1, e^{2\pi i/r} z_2) \sim (z_1, z_2)$ in the complex representation of the S^3 -sphere, $|z_1|^2 + |z_2|^2 = 1$.

The function $\gamma^{(r)}(z, m; p, q)$ looks rather different from $\Gamma(z; p, q)$, since it involves three bases p^r, q^r, pq and the discrete variable m . Let us show that, in fact, it is nothing else than a special product of the standard elliptic gamma functions with the bases p^r and q^r . Consider the double elliptic gamma function $\Gamma(z; p, q, t)$ with a special choice of the third base parameter $t = pq$. With its help, we can write

$$\begin{aligned} \gamma^{(r)}(z, m; p, q) &= \frac{\Gamma(q^r z p^m; p^r, q^r, pq)}{\Gamma(z p^m; p^r, q^r, pq)} \frac{\Gamma(p^r z q^{r-m}; p^r, q^r, pq)}{\Gamma(z q^{r-m}; p^r, q^r, pq)} \\ &= \frac{\Gamma((pq)^m q^{r-m} z; p^r, q^r, pq)}{\Gamma(q^{r-m} z; p^r, q^r, pq)} \frac{\Gamma((pq)^{r-m} p^m z; p^r, q^r, pq)}{\Gamma(p^m z; p^r, q^r, pq)}. \end{aligned} \quad (18)$$

The latter relation yields the product of two Pochhammer-type symbols built out of the elliptic gamma function with the bases p^r and q^r . For $0 \leq m \leq r-1$ we obtain the expression

$$\gamma^{(r)}(z, m; p, q) = \prod_{k=0}^{m-1} \Gamma(q^{r-m} z (pq)^k; p^r, q^r) \prod_{k=0}^{r-m-1} \Gamma(p^m z (pq)^k; p^r, q^r), \quad (19)$$

for $m < 0$ we have

$$\gamma^{(r)}(z, m; p, q) = \frac{\prod_{k=0}^{r-m-1} \Gamma(p^m z (pq)^k; p^r, q^r)}{\prod_{k=1}^{-m} \Gamma(q^{r-m} z (pq)^{-k}; p^r, q^r)}, \quad (20)$$

and for $m \geq r$,

$$\gamma^{(r)}(z, m; p, q) = \frac{\prod_{k=0}^{m-1} \Gamma(q^{r-m} z (pq)^k; p^r, q^r)}{\prod_{k=1}^{m-r} \Gamma(p^m z (pq)^{-k}; p^r, q^r)}. \quad (21)$$

The second order elliptic gamma function is related to superconformal indices in six dimensional field theories. Therefore it is natural to expect that there exists a natural physical meaning of the rarified elliptic gamma function from the point of view of compactification of six dimensional theories to lens spaces [3, 11].

From (19) and (21) it follows that for $r = 1, m = 0$ and $r = 1, m = 1$ we have the standard elliptic gamma function

$$\gamma^{(1)}(z, 0; p, q) = \gamma^{(1)}(z, 1; p, q) = \Gamma(z; p, q).$$

These equalities are related to the following factorized representation of the elliptic gamma function

$$\Gamma(z; p, q) = \prod_{j=0}^{\infty} \frac{1 - z^{-1}(pq)^j}{1 - z(pq)^j} \prod_{j,k=0}^{\infty} \frac{1 - z^{-1}(pq)^{j+1} p^{k+1}}{1 - z(pq)^j p^{k+1}} \frac{1 - z^{-1}(pq)^{j+1} q^{k+1}}{1 - z(pq)^j q^{k+1}}. \quad (22)$$

For $r = 1$ and $m \neq 0$ one can deduce directly from the definition (15) the recurrence relation

$$\gamma^{(1)}(z, m+1; p, q) = \frac{\theta(zp^m; pq)}{\theta(zq^{-m}; pq)} \gamma^{(1)}(z, m; p, q), \quad (23)$$

or the general relation

$$\begin{aligned} \gamma^{(1)}(z, m; p, q) &= \theta(z; pq|p)_m \theta(qz; pq|q)_{-m} \Gamma(z; p, q) \\ &= \left(-\frac{\sqrt{pq}}{z} \right)^{\frac{m(m-1)}{2}} \left(\frac{q}{p} \right)^{\frac{m(m-1)(2m-1)}{12}} \Gamma(z; p, q), \quad m \in \mathbb{Z}. \end{aligned} \quad (24)$$

As a result, the normalization condition $\Gamma(\sqrt{pq}; p, q) = 1$ is replaced by a more complicated relation

$$\gamma^{(1)}(\sqrt{pq}, m; p, q) = (-1)^{\frac{m(m-1)}{2}} (qp^{-1})^{\frac{m(m-1)(2m-1)}{12}}.$$

Instead of the exact (p, q) -permutational symmetry one now has

$$\gamma^{(r)}(z, m; p, q) = \gamma^{(r)}(z, r-m; q, p). \quad (25)$$

The $\gamma^{(r)}$ -function has an important quasiperiodicity property

$$\begin{aligned} \frac{\gamma^{(r)}(z, m+kr; p, q)}{\gamma^{(r)}(z, m; p, q)} &= \prod_{l=0}^{k-1} \frac{\theta(zp^{m+lr}; pq)}{\theta(zq^{-m-lr}; pq)} \\ &= \left(-\frac{\sqrt{pq}}{z} \right)^{mk+r\frac{k(k-1)}{2}} \left(\frac{q}{p} \right)^{\frac{1}{2}m^2k+mr\frac{k(k-1)}{2}+r^2\frac{k(k-1)(2k-1)}{12}}, \quad k \in \mathbb{Z}. \end{aligned} \quad (26)$$

Since any integer m can be represented in the form $m+kr$ with $0 \leq m \leq r-1$, $k \in \mathbb{Z}$, formulae (19) and (26) provide general representation of the $\gamma^{(r)}(z, m; p, q)$ -function as a product of elliptic gamma functions with the bases p^r and q^r up to some (cumbersome, but simple) exponential factor.

The inversion relation has the form

$$\gamma^{(r)}(z, m; p, q) \gamma^{(r)}\left(\frac{pq}{z}, r-m; p, q\right) = 1, \quad (27)$$

which is proved by using the definition (15) and formula (7). The most elementary equations for this function have the form

$$\begin{aligned} \gamma^{(r)}(qz, m+1; p, q) &= \theta(zp^m; p^r) \gamma^{(r)}(z, m; p, q), \\ \gamma^{(r)}(pz, m-1; p, q) &= \theta(zq^{r-m}; q^r) \gamma^{(r)}(z, m; p, q). \end{aligned} \quad (28)$$

Let us normalize the $\gamma^{(r)}(z, m; p, q)$ -function as follows

$$\Gamma^{(r)}(z, m; p, q) := \left(-\frac{z}{\sqrt{pq}} \right)^{\frac{m(m-1)}{2}} \left(\frac{p}{q} \right)^{\frac{m(m-1)(2m-1)}{12}} \gamma^{(r)}(z, m; p, q). \quad (29)$$

We call this function *the rarefied elliptic gamma function*. Its poles and zeros lie at the same points as in (16). For $r = 1$, independently on the value of $m \in \mathbb{Z}$, one has the equality

$$\Gamma^{(1)}(z, m; p, q) = \Gamma(z; p, q),$$

which is the source for the normalizing multiplier choice in (29).

The discrete variable quasiperiodicity takes now the form

$$\frac{\Gamma^{(r)}(z, m + kr; p, q)}{\Gamma^{(r)}(z, m; p, q)} = \left[\left(-\frac{z}{\sqrt{pq}} \right)^{(2m+rk)k} \left(\frac{p}{q} \right)^{mk(m+rk) + \frac{rk(2rk^2-1)}{6}} \right]^{\frac{r-1}{2}}, \quad k \in \mathbb{Z}. \quad (30)$$

The (p, q) -permutational symmetry transformation gets simplified to

$$\Gamma^{(r)}(z, m; p, q) = \Gamma^{(r)}(z, -m; q, p), \quad (31)$$

i.e. the permutation of p and q is equivalent to the change of sign $m \rightarrow -m$. The inversion relation also takes a natural compact form

$$\Gamma^{(r)}(z, m; p, q) \Gamma^{(r)}\left(\frac{pq}{z}, -m; p, q\right) = 1. \quad (32)$$

As a consequence one has the following relations

$$\Gamma^{(r)}(\sqrt{pq}, m; p, q) \Gamma^{(r)}(\sqrt{pq}, -m; p, q) = 1$$

and

$$\Gamma^{(r)}(\sqrt{pq}, 0; p, q) = 1, \quad (33)$$

which can be used as a normalization condition. For computing the residues we shall need the following limiting relation

$$\lim_{z \rightarrow 1} (1 - z) \Gamma^{(r)}(z, 0; p, q) = \lim_{z \rightarrow 1} (1 - z) \gamma^{(r)}(z, 0; p, q) = \frac{1}{(p^r; p^r)_\infty (q^r; q^r)_\infty}, \quad (34)$$

which is easily established from the representation (19), relation (9), and the identity $\prod_{k=1}^{r-1} \Gamma((pq)^k; p^r, q^r) = 1$.

The elementary recurrence relations take the form

$$\Gamma^{(r)}(qz, m + 1; p, q) = (-z)^m p^{\frac{m(m-1)}{2}} \theta(zp^m; p^r) \Gamma^{(r)}(z, m; p, q), \quad (35)$$

$$\Gamma^{(r)}(pz, m - 1; p, q) = (-z)^{-m} q^{\frac{m(m+1)}{2}} \theta(zq^{-m}; q^r) \Gamma^{(r)}(z, m; p, q) \quad (36)$$

or

$$\begin{aligned} \Gamma^{(r)}(q^{-1}z, m - 1; p, q) &= \left(-\frac{z}{pq} \right)^{-m} p^{-\frac{m(m+1)}{2}} \frac{\Gamma^{(r)}(z, m; p, q)}{\theta(z^{-1}qp^{1-m}; p^r)}, \\ \Gamma^{(r)}(p^{-1}z, m + 1; p, q) &= \left(-\frac{z}{pq} \right)^m q^{-\frac{m(m-1)}{2}} \frac{\Gamma^{(r)}(z, m; p, q)}{\theta(z^{-1}pq^{1+m}; q^r)}. \end{aligned}$$

Note that equations (35) and (36) do not determine uniquely the function $\Gamma^{(r)}(z, m; p, q)$. The general solution of these equations has the form $\Gamma^{(r)}(z, m; p, q) \varphi_m(z)$, where the functions $\varphi_m(z)$ satisfy the recurrences

$$\varphi_{m+1}(qz) = \varphi_m(z), \quad \varphi_{m-1}(pz) = \varphi_m(z).$$

Resolution of the first equation yields $\varphi_m(z) = \varphi_0(q^{-m}z)$ for arbitrary function $\varphi_0(z)$. The second equation yields $\varphi_0(pqz) = \varphi_0(z)$, i.e. $\varphi_0(z)$ is an elliptic function of z with the modular parameter pq , i.e.

$$\varphi_0(z) = \prod_{k=1}^K \frac{\theta(\alpha_k z; pq)}{\theta(\beta_k z; pq)}, \quad \prod_{k=1}^K \alpha_k = \prod_{k=1}^K \beta_k,$$

for some integer $K = 0, 2, 3, \dots$, which is called the order of an elliptic function. Here the parameters $\alpha_k, \beta_k \in \mathbb{C}^*$ are arbitrary (up to one constraint) variables forming the divisor set of $\varphi_0(z)$. So, the space of solutions of interest has a functional freedom. However, the quasiperiodicity condition (30) removes this freedom. Indeed, as a consequence of (30) one gets the additional constraint $\varphi_0(q^r z) = \varphi_0(z)$. For incommensurate p and q , all indicated restrictions for $\varphi_0(z)$ can be satisfied only by a constant, $\varphi_0(z) = \text{const}$, which is fixed by the normalization condition (33).

For $0 \leq m \leq r-1$, one can write

$$\begin{aligned} \Gamma^{(r)}(z, m; p, q) &= (-z)^{\frac{m(m-1)}{2}} p^{\frac{m(m-1)(m-2)}{6}} q^{-\frac{m(m^2-1)}{6}} \\ &\times \prod_{k=0}^{m-1} \Gamma(q^{r-m} z (pq)^k; p^r, q^r) \prod_{k=0}^{r-m-1} \Gamma(p^m z (pq)^k; p^r, q^r). \end{aligned} \quad (37)$$

This relation together with (30) expresses $\Gamma^{(r)}(z, m; p, q)$ -function with arbitrary m as a product of ordinary elliptic gamma functions, since any m can be represented in the form $\ell + kr$ with $0 \leq \ell \leq r-1$, $k \in \mathbb{Z}$. Therefore all interesting integrals constructed from the $\Gamma^{(r)}(z, m; p, q)$ -function are expected to be directly related to some standard elliptic hypergeometric integrals [12, 17].

4. A RAREFIED ELLIPTIC BETA INTEGRAL

We define a kernel of a would be elliptic beta sum-integral

$$\Delta(z, m; t_a, n_a | p, q) := \frac{\prod_{a=1}^6 \Gamma^{(r)}(t_a z^{\pm 1}, n_a \pm m; p, q)}{\Gamma^{(r)}(z^{\pm 2}, \pm 2m; p, q)}, \quad (38)$$

where we adopted the compact notation

$$\Gamma^{(r)}(tz^{\pm 1}, n \pm m; p, q) := \Gamma^{(r)}(tz, n + m; p, q) \Gamma^{(r)}(tz^{-1}, n - m; p, q). \quad (39)$$

This function has the permutational symmetry

$$\Delta(z, m; t_a, n_a | p, q) = \Delta(z, -m; t_a, -n_a | q, p). \quad (40)$$

From now on, for brevity we drop the bases p and q from the notation for integrands and rarefied elliptic gamma functions.

It is not difficult to derive the following equations

$$\frac{\Delta(pz, m-1; t_a, n_a)}{\Delta(z, m; t_a, n_a)} = h_1(z, m), \quad \frac{\Delta(qz, m+1; t_a, n_a)}{\Delta(z, m; t_a, n_a)} = h_2(z, m), \quad (41)$$

where

$$h_1(z, m) = c_1 \prod_{a=1}^6 \frac{\theta(t_a z q^{-n_a-m}; q^r)}{\theta(t_a^{-1} p q z q^{n_a-m}; q^r)} \frac{\theta((pqz)^2 q^{-2m}; q^r)}{\theta(z^2 q^{-2m}; q^r)} \quad (42)$$

and

$$h_2(z, m) = c_2 \prod_{a=1}^6 \frac{\theta(t_a z p^{n_a+m}; p^r)}{\theta(t_a^{-1} p q z p^{-n_a+m}; p^r)} \frac{\theta((pqz)^2 p^{2m}; p^r)}{\theta(z^2 p^{2m}; p^r)}, \quad (43)$$

with $c_{1,2}$ being some constants independent on z . One can check that

$$\frac{h_1(q^r z, m)}{h_1(z, m)} = q^{2 \sum_{a=1}^6 n_a} \frac{(pq)^2}{\prod_{a=1}^6 t_a^2}, \quad \frac{h_2(p^r z, m)}{h_2(z, m)} = p^{-2 \sum_{a=1}^6 n_a} \frac{(pq)^2}{\prod_{a=1}^6 t_a^2}. \quad (44)$$

Therefore, imposing the balancing condition

$$\prod_{a=1}^6 t_a = pq, \quad \sum_{a=1}^6 n_a = 0, \quad (45)$$

we obtain

$$h_1(q^r z, m) = h_1(z, m), \quad h_2(p^r z, m) = h_2(z, m). \quad (46)$$

Denoting $q = e^{2\pi i \sigma}$, $p = e^{2\pi i \tau}$, we see that $h_1(e^{2\pi i u}, m)$ becomes an elliptic function of u of the order 10 with the periods 1 and $r\sigma$, while $h_2(e^{2\pi i u}, m)$ becomes an elliptic function of u of the order 10 with the periods 1 and $r\tau$. Note that one could fix the balancing condition for the continuous parameters as $\prod_{a=1}^6 t_a = -pq$, which also leads to elliptic functions, but, similar to the standard elliptic beta integral case [12], the choice (45) is a distinguished one.

After imposing the balancing condition (45) the constants $c_{1,2}$ simplify to

$$c_1 = c_2 = pq. \quad (47)$$

Moreover, it is easy to check that now $\Delta(t_a, n_a; z, m)$ becomes a periodic function of m :

$$\Delta(z, m+r; t_a, n_a) = \Delta(z, m; t_a, n_a), \quad (48)$$

since all quasiperiodicity factors emerging from the relation (30) cancel out. Using this fact we repeat r times the recurrence relations (41) and obtain

$$\Delta(p^r z, m; t_a, n_a) = \prod_{k=0}^{r-1} h_1(p^k z, m-k) \Delta(z, m; t_a, n_a), \quad (49)$$

$$\Delta(q^r z, m; t_a, n_a) = \prod_{k=0}^{r-1} h_2(q^k z, m+k) \Delta(z, m; t_a, n_a). \quad (50)$$

Therefore, the function $\Delta(z, m; t_a, n_a)$ is a solution of a finite-difference equation of the first order with the coefficient given by a particular elliptic function of the order $10r$.

We remind [17] that in the multiplicative notation the contour integral $\int_C \Delta(z) dz/z$ is called an elliptic hypergeometric integral, if $\Delta(z)$ satisfies the first order q -difference

equation $\Delta(qz) = h(z)\Delta(z)$ with a p -periodic (i.e., elliptic) function $h(z)$, $h(pz) = h(z)$. Therefore, if we consider a contour integral of our Δ -function, by definition we obtain a standard elliptic hypergeometric integral with the bases p and q replaced by p^r and q^r , respectively. If we further sum over m we get an elliptic hypergeometric “sum-integral”. We call such objects as rarefied elliptic hypergeometric integrals, since they are represented by sums of the standard elliptic hypergeometric integrals whose parameters are fixed in a particular way using the powers of $p^{1/r}$ and $q^{1/r}$ (in the notation of formula (12)), which justifies the use of the term “rarefied”. Using representations (19)-(21) one can rewrite the function $\Delta(z, m; t_a, n_a)$ as a ratio of standard elliptic gamma functions with the bases p^r and q^r , however, the resulting expressions are cumbersome and we do not present them here.

A rarefied analogue of the elliptic beta integral (12) has the following form.

Theorem 2. *Take eight continuous parameters $t_1, \dots, t_6, p, q \in \mathbb{C}$ and six discrete parameters $n_1, \dots, n_6 \in \mathbb{Z}$ and impose the constraints $|t_a|, |p|, |q| < 1$ and the balancing condition*

$$\prod_{a=1}^6 t_a = pq, \quad \sum_{a=1}^6 n_a = 0.$$

Then

$$\frac{(p^r; p^r)_\infty (q^r; q^r)_\infty}{4\pi i} \sum_{m=0}^{r-1} \int_{\mathbb{T}} \Delta(z, m; t_a, n_a) \frac{dz}{z} = \prod_{1 \leq a < b \leq 6} \Gamma^{(r)}(t_a t_b, n_a + n_b; p, q), \quad (51)$$

where \mathbb{T} is the unit circle of positive orientation.

The constraints $|t_a| < 1$ can be relaxed by replacing for each fixed m the integration contour \mathbb{T} by its deformations C_m such that they separate sequences of the Δ -function poles converging to zero (they form geometric progressions with different nomes) from their partners diverging to infinity. The conditions of existence of such contours are complicated, they impose certain constraints on the parameters and require thorough considerations. In this case, evidently, one cannot permute the summation over m and integration over z in (51). Note also that, because of the periodicity (48), the sum over $m = 0, 1, \dots, r-1$ can be replaced by the sum over any r sequential values of the integer m . According to the discussion given above, on the left-hand side of identity (51) we have a sum of r ordinary elliptic hypergeometric integrals.

The proof of relation (51) will be given in the next section in a substantially more general situation together with a comparison with the results of [8].

5. A C_n RAREFIED ELLIPTIC BETA INTEGRAL OF TYPE I

We define the kernel of the type I multiple elliptic beta sum-integral for the root system C_n as

$$\begin{aligned} \Delta_I^{(r)}(z_j, m_j; t_a, n_a) &:= \prod_{1 \leq j < k \leq n} \frac{1}{\Gamma^{(r)}(z_j^{\pm 1} z_k^{\pm 1}, \pm m_j \pm m_k; p, q)} \\ &\times \prod_{j=1}^n \frac{\prod_{a=1}^{2n+4} \Gamma^{(r)}(t_a z_j^{\pm 1}, n_a \pm m_j; p, q)}{\Gamma^{(r)}(z_j^{\pm 2}, \pm 2m_j; p, q)}, \quad t_a, z_j \in \mathbb{C}^*, n_a, m_j \in \mathbb{Z}, \end{aligned} \quad (52)$$

and impose the balancing condition

$$\prod_{a=1}^{2n+4} t_a = pq, \quad \sum_{a=1}^{2n+4} n_a = 0. \quad (53)$$

Association with the root system C_n comes from the fact that the denominator of the ratio of the products of rarefied elliptic gamma functions in (52) can be formally written as $\prod_{\alpha \in R(C_n)} \Gamma^{(r)}(e^{u\alpha}, m\alpha; p, q)$, where $\alpha \in \{\pm e_i \pm e_j, \pm 2e_i\}_{i,j=1,\dots,n}$ are the roots of the root system $R(C_n)$ with e_i being the standard euclidean basis vectors of \mathbb{R}^n and $z_i := e^{ue_i}$, $m_i := me_i$ for some formal variables $u \in \mathbb{C}$ and $m \in \mathbb{Z}$.

Theorem 3. Take $2n + 4$ continuous and discrete parameters $t_a \in \mathbb{C}^*$, $n_a \in \mathbb{Z}$, $a = 1, \dots, 2n+4$, and bases $p, q \in \mathbb{C}$ satisfying the restrictions $|p|, |q|, |t_a| < 1$ and the balancing condition (53). Denote

$$\kappa_n^{(r)} = \frac{(p^r; p^r)_\infty^n (q^r; q^r)_\infty^n}{(4\pi i)^n n!}.$$

Then

$$\kappa_n^{(r)} \sum_{m_1, \dots, m_n=0}^{r-1} \int_{\mathbb{T}^n} \Delta_I^{(r)}(z_j, m_j; t_a, n_a) \prod_{j=1}^n \frac{dz_j}{z_j} = \prod_{1 \leq a < b \leq 2n+4} \Gamma^{(r)}(t_a t_b, n_a + n_b; p, q), \quad (54)$$

where \mathbb{T} is the unit circle of positive orientation.

Proof. For proving this identity we remove parameters t_{2n+4} and n_{2n+4} using the balancing constraint. For that we denote

$$A := \prod_{a=1}^{2n+3} t_a = \frac{pq}{t_{2n+4}}, \quad N := \sum_{a=1}^{2n+3} n_a = -n_{2n+4},$$

and apply the inversion relation for the rarefied elliptic gamma function involving parameter t_{2n+4} . Now we divide the left-hand side of equality (54) by the expression on the right-hand side and rewrite it as

$$I(t_1, \dots, t_{2n+3}, n_1, \dots, n_{2n+3}) := \sum_{m=0}^{r-1} \int_{\mathbb{T}} \rho(z_j, m_j; t_a, n_a) \prod_{j=1}^n \frac{dz_j}{z_j} = \frac{1}{\kappa_n^{(r)}}, \quad (55)$$

where

$$\begin{aligned} \rho(z_j, m_j; t_a, n_a) := & \prod_{1 \leq j < k \leq n} \frac{1}{\Gamma^{(r)}(z_j^{\pm 1} z_k^{\pm 1}, \pm m_j \pm m_k)} \\ & \times \prod_{j=1}^n \frac{\prod_{a=1}^{2n+3} \Gamma^{(r)}(t_a z_j^{\pm 1}, n_a \pm m_j)}{\Gamma^{(r)}(z_j^{\pm 2}, \pm 2m_j) \Gamma^{(r)}(A z_j^{\pm 1}, N \pm m_j)} \frac{\prod_{a=1}^{2n+3} \Gamma^{(r)}(A t_a^{-1}, N - n_a)}{\prod_{1 \leq a < b \leq 2n+3} \Gamma^{(r)}(t_a t_b, n_a + n_b)}. \end{aligned} \quad (56)$$

This function is periodic in all discrete variables m_j , $j = 1, \dots, n$, and n_a , $a = 1, \dots, 2n+3$:

$$\rho(z_j, \dots, m_k + r, \dots) = \rho(z_j, \dots, n_b + r, \dots) = \rho(z_j, m_j; t_a, n_a),$$

which is a very important property following from a lengthy cancellation of the complicated quasiperiodicity multipliers generated by the rarefied elliptic gamma functions.

Consider the divisor of (56) considered as a function of z_ℓ . Due to the property

$$\Gamma^{(r)}(z, m) \Gamma^{(r)}(z^{-1}, -m) = \frac{(pq)^{\frac{m(m+1)}{2}}}{\theta(zq^{-m}; q^r) \theta(z^{-1}p^{-m}; p^r)},$$

it does not contain poles whose positions do not depend on t_a (at $z_\ell = 0$ one has an essential singularity). The t_a -independent zeros do not play any role in the following considerations and we skip them. As to the t_a -dependent poles and zeros, a naive consideration shows that the function $\rho(z_\ell, \dots)$ has sequences of poles converging to $z_\ell = 0$ for any ℓ by the points of the sets

$$P_{in}^A = \{t_a q^k p^{n_a - m_\ell + k + rj}\}, \quad P_{in}^B = \{t_a q^{r - n_a + m_\ell + k + rj} p^k\}$$

with $a = 1, \dots, 2n+4$ and $j, k \in \mathbb{Z}_{\geq 0}$, and going to infinity along the sets

$$P_{out}^A = \{t_a^{-1} q^{-k} p^{-n_a - m_\ell - k - rj}\}, \quad P_{out}^B = \{t_a^{-1} q^{n_a + m_\ell - k - r(j+1)} p^{-k}\},$$

which are not identically $z \rightarrow 1/z$ reciprocal to P_{in} . Zeros of this function converge to $z_\ell = 0$ for any ℓ by the point sets

$$Z_{in}^A = \{t_a^{-1} q^{k+1} p^{-n_a - m_\ell + k + 1 + r(j+1)}\}, \quad Z_{in}^B = \{t_a^{-1} q^{n_a + m_\ell + k + 1 + rj} p^{k+1}\}$$

with $a = 1, \dots, 2n+4$ and $j, k \in \mathbb{Z}_{\geq 0}$, and go to infinity along the point sets

$$Z_{out}^A = \{t_a q^{-k-1} p^{n_a - m_\ell - k - 1 - r(j+1)}\}, \quad Z_{out}^B = \{t_a q^{-n_a + m_\ell - k - 1 - rj} p^{-k-1}\},$$

which are not identically $z \rightarrow 1/z$ reciprocal to Z_{in} .

However, the structure of poles and zeros is rather complicated and it may happen that in the sets indicated above positions of some poles and zeros coincide and, actually, both are absent. First, we assume that the parameters t_a and bases p, q are multiplicatively incommensurate, i.e. $t_a^n t_b^m p^k q^l \neq 1$ for $n, m, k, l \in \mathbb{Z}$, which guarantees that all poles and zeros are simple. Then, equating positions of poles and zeros (with j, k replaced by j', k'), we find that nontrivial cancellations may exist only if $j = j'$ and either

$$rj + k + k' = -n_a + m_\ell - 1, \quad (57)$$

if there are intersecting points in P_{in}^A and Z_{out}^B , or

$$rj + k + k' = n_a - m_\ell - r - 1, \quad (58)$$

if P_{in}^B and Z_{out}^A overlap, or

$$rj + k + k' = -n_a - m_\ell - 1, \quad (59)$$

if P_{out}^A and Z_{in}^B overlap, or

$$rj + k + k' = n_a + m_\ell - r - 1, \quad (60)$$

if P_{out}^B and of Z_{in}^A overlap. Let us denote as p_{in}^{max} the maximal possible absolute value of the pole positions in some indicated subset of P_{in} and as p_{out}^{min} the minimal possible absolute value of the pole positions in some indicated subset of P_{out} .

Remind that $j, k, k' \geq 0$, but n_a and m_ℓ can take arbitrary integer values. The periodicity (57) means that the poles of the ρ -function form a periodic lattice in n_a and m_ℓ and the above equations for j, k, k' always have solutions for sufficiently large $|n_a|$ and $|m|$, in which case a part of the poles is cancelled by zeros. Therefore, without loss of the generality, we can restrict the values of m and n_a to

$$0 \leq m \leq r - 1, \quad -r < n_a < r$$

(this can be done simply by the shifts $n_a \rightarrow n_a \pm r$ and $n_b \rightarrow n_b \mp r$, $b > a$, as soon as one gets $|\sum_{k=1}^a n_a| \geq r$ for $a = 1, \dots, 2n + 3$. As a result, we have $-2r + 1 < n_a - m_\ell < r$, which means that equation (58) has not solutions and $p_{in,B}^{max} = \max |qt_a|$.

Suppose now that $n_a - m_\ell \geq 0$. Then equation (57) has not solutions and $p_{in,A}^{max} = \max |t_a|$, which is reached only for $m_\ell = n_a$ with the corresponding value of a . Let now $m_\ell \geq n_a + 1$. Then equation (57) may have nontrivial solutions. For $j = 0$ one has $k, k' = 0, 1, \dots, m_\ell - n_a - 1$, so that $p_{in,A,j=0}^{max} = \max |qt_a|$. For $j = 1$ and $0 < m_\ell - n_a < r + 1$ there are no solutions and the maximal absolute value of the corresponding pole positions is $\max |t_a|$. For $j = 1$ and $m_\ell - n_a \geq r + 1$ (which can be satisfied only for $r > 2$) the solution is $k, k' = 0, 1, \dots, m_\ell - n_a - 1 - r$, and the top possible pole position has the absolute value $\max |qt_a|$. So, $p_{in,A,j=1}^{max} = \max |t_a|$. The poles in P_{in}^A with $j > 1$ and $m_\ell \geq n_a + 1$ have $p_{in,A,j>1}^{max} = \max |p^2 t_a|$. So, for $|t_a| < 1$ all the poles from P_{in} lie inside \mathbb{T} .

Similar situation takes place for the poles P_{out} and zeros Z_{in} . Indeed, for $n_a + m_\ell \geq 0$ there are no solutions of equation (59) with $p_{out,A}^{min} = \min |t_a^{-1}|$, which is reached for $m_\ell = -n_a$ with the corresponding value of a . For $m_\ell < -n_a$ the solution of (59) is $j = 0$ and $k, k' = 0, \dots, -n_a - m_\ell - 1$ with $p_{out,A,j=0}^{min} = \min |t_a^{-1} q^{-1}|$. The poles with $j > 0$ have $p_{out,A,j>0}^{min} = \min |t_a^{-1} p^{-1}|$.

Consider now P_{out}^B . For $n_a + m_\ell \leq r$ equation (60) has not solutions and $p_{out,B}^{min} = \min |t_a^{-1}|$, which is reached for $m_\ell = r - n_a$ with the corresponding value of a . For $n_a + m_\ell > r$ (which can be satisfied only for $r > 2$) the solutions of (60) have the form $j = 0$ and $k, k' = 0, \dots, n_a + m_\ell - r - 1$ with $p_{out,B,j=0}^{min} = \min |t_a^{-1} p^{-1}|$. Finally, for $j > 0$ equation (60) has not solutions and one has $p_{out,B,j>0}^{min} = \min |t_a^{-1} q^{-2}|$. So, all the poles from P_{out} lie outside \mathbb{T} for $|t_a| < 1$.

To conclude, if we impose the constraint $\max |t_a| < 1$, then all poles from P_{in} and P_{out} are pushed inside and outside of \mathbb{T} , respectively. It is exactly this property (which is established after a rather neat analysis of the structure of ρ -function divisor points) that determines the choice of \mathbb{T} as the integration contour in formula (54).

Now we prove the following finite-difference equation:

$$\begin{aligned} & \rho(z_j, m_j; pt_1, t_2, \dots, n_1 - 1, n_2, \dots) - \rho(z_j, m_j; t_a, n_a) \\ &= \sum_{k=1}^n \left(g_k(\dots, p^{-1}z_k, \dots, m_k + 1, \dots; t_a, n_a) - g_k(z_j, m_j; t_a, n_a) \right), \end{aligned} \quad (61)$$

where

$$\begin{aligned} g_k(z_j, m_j; t_a, n_a) &= \rho(z_j, m_j; t_a, n_a) \prod_{\ell=1, \ell \neq k}^n \frac{\theta(t_1 z_\ell^{\pm 1} q^{-n_1 \mp m_\ell}; q^r)}{\theta(z_k z_\ell^{\pm 1} q^{-m_k \mp m_\ell}; q^r)} \\ &\times \frac{\prod_{a=1}^{2n+3} \theta(t_a z_k q^{-n_a - m_k}; q^r)}{\prod_{a=2}^{2n+3} \theta(t_1 t_a q^{-n_1 - n_a}; q^r)} \frac{\theta(t_1 A q^{-n_1 - N}; q^r)}{\theta(z_k^2 q^{-2m_k}, A z_k q^{-N - m_k}; q^r)} \frac{t_1 q^{m_k}}{z_k q^{n_1}}. \end{aligned} \quad (62)$$

Dividing this equation by $\rho(z_j, m_j; t_a, n_a)$ we come to the following identity

$$\begin{aligned} & \prod_{j=1}^n \frac{\theta(t_1 z_j^{\pm 1} q^{-n_1 \mp m_j}; q^r)}{\theta(A z_j^{\pm 1} q^{-N \mp m_j}; q^r)} \prod_{a=2}^{2n+3} \frac{\theta(A t_a^{-1} q^{-N + n_a}; q^r)}{\theta(t_1 t_a q^{-n_1 - n_a}; q^r)} - 1 \\ &= \frac{t_1 q^{-n_1} \theta(t_1 A q^{-n_1 - N}; q^r)}{\prod_{a=2}^{2n+3} \theta(t_1 t_a q^{-n_1 - n_a}; q^r)} \sum_{k=1}^n \frac{q^{m_k}}{z_k \theta(z_k^2 q^{-2m_k}; q^r)} \prod_{j=1, j \neq k}^n \frac{\theta(t_1 z_j^{\pm 1} q^{-n_1 \mp m_j}; q^r)}{\theta(z_k z_j^{\pm 1} q^{-m_k \mp m_j}; q^r)} \\ &\times \left(\frac{z_k^{2n+2} \prod_{a=1}^{2n+3} \theta(t_a z_k^{-1} q^{-n_a + m_k}; q^r)}{q^{m_k(2n+2)} \theta(A z_k^{-1} q^{-N + m_k}; q^r)} - \frac{\prod_{a=1}^{2n+3} \theta(t_a z_k q^{-n_a - m_k}; q^r)}{\theta(A z_k q^{-N - m_k}; q^r)} \right). \end{aligned} \quad (63)$$

The shifts $z_j \rightarrow z_j q^{m_j}$ and $t_a \rightarrow t_a q^{n_a}$ remove completely the discrete variables m_j and n_a from (63) and we obtain precisely the elliptic functions identity established in [12] in the proof of the type I van Diejen-Spiridonov integral (with p and q replaced by p^r and q^r).

We integrate now equation (61) over the multi-contour \mathbb{T}^n and sum over all m_j from 0 to $r - 1$. It can be checked that g_k -functions are periodic with respect to the shifts $m_j \rightarrow m_j + r$. Therefore we obtain

$$\begin{aligned} & I(pt_1, t_2, \dots, t_{2n+3}, n_1 - 1, n_2, \dots, n_{2n+3}) - I(t_a, n_a) \\ &= \sum_{m_1, \dots, m_n=0}^{r-1} \sum_{\ell=1}^n \left(\int_{\mathbb{T}^{\ell-1} \times (p^{-1}\mathbb{T}) \times \mathbb{T}^{n-\ell}} - \int_{\mathbb{T}^n} \right) g_\ell(z_j, m_j; t_a, n_a) \prod_{j=1}^n \frac{dz_j}{z_j}, \end{aligned} \quad (64)$$

where $p^{-1}\mathbb{T}$ denotes the contour obtained from \mathbb{T} after blowing it by the factor p^{-1} .

The divisor points of the functions g_ℓ (62) in the variable z_ℓ are determined by the following factor (theta functions were absorbed into the gamma functions by appropriate shifts of the arguments):

$$\begin{aligned} & \prod_{a=1}^{2n+3} \left[\Gamma^{(r)}(pt_a z_\ell, n_a + m_\ell - 1) \Gamma^{(r)}(t_a z_\ell^{-1}, n_a - m_\ell) \right] \\ & \times \Gamma^{(r)}(t_{2n+4} z_\ell, n_{2n+4} + m_\ell) \Gamma^{(r)}(p^{-1} t_{2n+4} z_\ell^{-1}, n_{2n+4} - m_\ell + 1). \end{aligned}$$

Comparing with the previous analysis of the divisor of the ρ -function, we see that the equations (57) and (58) are preserved for P_{in} poles associated with t_a , $a = 1, \dots, 2n+3$, whereas, vice versa, equations (59) and (60) remain the same for P_{out} poles associated with t_{2n+4} . The sum $rj+k+k'$ is equal to $-n_{2n+4}+m_\ell-2$ or $n_{2n+4}-m_\ell-r$ for the analogues of (57) and (58) with $a = 2n+4$, respectively, and to $-n_a-m_\ell$ or $n_a+m_\ell-r-2$ for the analogues of (59) and (60) with $a = 1, \dots, 2n+3$. As a result of such changes we find that $p_{in}^{max} = \max\{|t_a|, |p^{-1}t_{2n+4}|\}$ and $p_{out}^{min} = \min\{|t_{2n+4}^{-1}|, |p^{-1}t_a|\}$, where $a = 1, \dots, 2n+3$.

Therefore, for $|t_a| < 1$, $a = 1, \dots, 2n+3$ and $|t_{2n+4}| < |p|$ the functions g_ℓ do not have poles in the annuli $1 \leq |z_\ell| \leq |p^{-1}|$. As a result, we can safely shrink the integration contour $p^{-1}\mathbb{T}$ to \mathbb{T} in (64) and obtain zero on the right-hand side, i.e. the equality

$$I(pt_1, t_1, \dots, n_1 - 1, n_2, \dots) = I(t_a, n_a). \quad (65)$$

Note that for the taken constraints on the parameters the contour \mathbb{T} is legitimate for both integrals on the left-hand side of (64), i.e. it separates relevant sets of poles.

Due to the incommensurability condition, the integral $I(t_a, n_a)$ is a meromorphic function of the parameters t_a . Therefore, equation (65) can be used for analytical continuation of $I(t_a, n_a)$ from the domain $|t_1|, |t_{2n+4}| < 1$ to $|p^k t_1|, |p^{-k} t_{2n+4}| < 1$ for any $k \in \mathbb{Z}$. Therefore, iterating (65) r times and using the periodicity property

$$I(\dots, n_{b-1}, n_b + r, n_{b+1}, \dots) = I(t_a, n_a), \quad b = 1, \dots, 2n+3,$$

following from the ρ -function periodicity in variables n_b , we obtain the equality

$$I(p^r t_1, t_2, \dots, n_a) = I(t_a, n_a). \quad (66)$$

Let us impose the additional constraint $|t_{2n+4}| < |q|$. Then we can permute bases p and q in the above considerations and obtain the equality

$$I(q^r t_1, t_2, \dots, -n_a) = I(t_a, -n_a). \quad (67)$$

However, our analysis is symmetric with respect to the reflections $n_a \rightarrow -n_a$. Therefore, we have

$$I(p^r t_1, t_2, \dots, n_a) = I(q^r t_1, t_2, \dots, n_a) = I(t_1, t_2, \dots, n_a).$$

Since our bases p and q are incommensurate, this means that $I(t_a, n_a)$ does not depend on the parameter t_1 and, by symmetry, on all parameters t_a . Substituting this condition into recursion (65), we find that, actually, $I(t_a, n_a)$ does not depend on n_a as well, i.e. it is a constant depending only on p, q, r and n , $I(t_a, n_a) = c(p, q, r, n)$. Let us compute this constant c .

For that we set $n_a = 0$, $a = 1, \dots, 2n+4$, and consider the limit $t_a t_{a+n} \rightarrow 1$, $a = 1, \dots, n$. Our analysis of the ρ -function divisor structure shows that in each summation over the discrete variables $0 \leq m_j \leq r-1$ there is one integral, corresponding to the value $m_j = 0$, for which the integration contour \mathbb{T} becomes pinched by $2n$ pairs of poles. The ρ -function contains the factor $1/\prod_{j=1}^n \Gamma^{(r)}(t_j t_{j+n}, 0)$ which vanishes unless it is cancelled by the residues of poles pinching the integration contour for all n integrals simultaneously.

Therefore, our problem reduces to the computation of the limit

$$\lim_{\substack{t_a t_{a+n} \rightarrow 1 \\ a=1, \dots, n}} \frac{1}{\prod_{1 \leq a < b \leq 2n+4} \Gamma^{(r)}(t_a t_b, 0)} \int_{\mathbb{T}^n} \prod_{1 \leq j < k \leq n} \frac{1}{\Gamma^{(r)}(z_j^{\pm 1} z_k^{\pm 1}, 0)} \prod_{j=1}^n \frac{\prod_{a=1}^{2n+4} \Gamma^{(r)}(t_a z_j^{\pm 1}, 0)}{\Gamma^{(r)}(z_j^{\pm 2}, 0)} \frac{dz_j}{z_j}.$$

Let us deform all \mathbb{T} to a contour C which crosses the poles $z_j = t_a$, $a = 1, \dots, 2n$, and does not touch other poles. Again, the result does not vanish only if we pick up residues for all variables z_j simultaneously. Whenever two different variables z_j pick up residues from identical pole positions, we get zero due to the functions $\Gamma^{(r)}(z_j^{\pm 1} z_k^{\pm 1}, 0)$ in the integrand's denominator. Therefore, we should consider only the residues for $z_j = t_j$ and their $n!$ permutations giving identical results. The residues for $z_j = t_{j+n}$ give the same result due to the symmetry of the integrand $z_j \rightarrow 1/z_j$, which results in the additional multiplier 2^n . So, the limit of interest is equal to

$$\begin{aligned} & 2^n n! (2\pi i)^n \lim_{\substack{t_a t_{a+n} \rightarrow 1 \\ a=1, \dots, n}} \frac{1}{\prod_{1 \leq a < b \leq 2n+4} \Gamma^{(r)}(t_a t_b, 0)} \\ & \times \lim_{z_j \rightarrow t_j} \prod_{j=1}^n \left(1 - \frac{t_j}{z_j}\right) \frac{\prod_{j=1}^n \prod_{a=1}^{2n+4} \Gamma^{(r)}(t_a z_j^{\pm 1}, 0)}{\prod_{1 \leq j < k \leq n} \Gamma^{(r)}(z_j^{\pm 1} z_k^{\pm 1}, 0) \prod_{j=1}^n \Gamma^{(r)}(z_j^{\pm 2}, 0)}. \end{aligned}$$

Applying the limiting relation (34) and verifying some tedious cancellations, we finally obtain

$$c(p, q, r, n) = \lim_{\substack{t_a t_{a+n} \rightarrow 1 \\ a=1, \dots, n}} I(t_j, n_j = 0) = \frac{(4\pi i)^n n!}{(p^r; p^r)_\infty^n (q^r; q^r)_\infty^n} = \frac{1}{\kappa_n^{(r)}},$$

as required.

Finally, by analytical continuation, we replace the restrictions $|t_{2n+4}| < |p|, |q|$ by $|t_{2n+4}| < 1$ and remove the incommensurability constraint $t_a^n t_b^m p^k q^l \neq 1$, $n, m, k, l \in \mathbb{Z}$ (still keeping $|t_a|, |p|, |q| < 1$). The theorem is proved. \square

Evidently, in the final result (54) one can relax restrictions for t_a parameter values by changing \mathbb{T} to a contour C such that it separates the poles P_{in} and P_{out} for all possible values of m_ℓ . However, the analysis of sufficient conditions for existence of such a contour is a complicated task and we do not consider it here.

Comparing the theorem proven above for $n = 1$ (i.e., the identity (51)) and the results of Kels [8], one can see that they differ in several ways. First, in [8] the function $[[m]] := m \bmod r$ is used which is not needed at all in our consideration, since all necessary finite-dimensional truncations are guaranteed by the r -periodicity in all discrete variables of the integrand. This makes our formulae analytical in the discrete variables n_a and $m := m_1$ in sharp difference from [8]. Second, more important, the integrand in (51) differs from

the one used in [8]. Indeed, in our case we can write

$$\begin{aligned} \Delta(t_a, n_a; z, m|p, q) &= \mu(t_a, n_a) \left(\frac{q}{p}\right)^{m^2} \frac{\prod_{a=1}^6 \gamma^{(r)}(t_a z^{\pm 1}, n_a \pm m; p, q)}{z^{2m} \gamma^{(r)}(z^{\pm 2}, \pm 2m; p, q)}, \\ \mu(t_a, n_a) &= \prod_{a=1}^6 t_a^{n_a(n_a-1)} \left(\frac{p}{q}\right)^{\frac{1}{3}n_a^3} p^{-n_a^2}, \end{aligned} \quad (68)$$

which is an explicit meromorphic function of $z \in \mathbb{C}^*$. The integrand of [8] has the same $\gamma^{(r)}$ -dependent part as above, up to the non-analytical dependence on the discrete variables, but the multiplier in front of it is substantially different. This follows from the fact that we use different normalizing factor for the rarefied elliptic gamma function (29) than in [8]. For instance, in (68) we have the z -dependent multiplier z^{-2m} for any admissible n_a , whereas it is easy to see that for $n_a = 0$ there is no any z -dependent multiplier in the integrand of [8]. Next, an analogue of the identity (63) in [8] contains fractional powers of z and it is different from our identity, which is the same as in [15]. Finally, the analysis of the divisor points of the integrands in our case is essentially more complicated, since we check case by case the possibilities of cancellation of poles and zeros, whereas in [8] the divisor was forced to take a particular form by the use of non-analytical functions of the discrete variables. Altogether, we see that our identity (51) critically differs from the one considered in [8].

6. A C_n RAREFIED ELLIPTIC BETA INTEGRAL OF TYPE II

The rarefied analogue of the type II van Diejen-Spiridonov integral [6] has the following form. For convenience we denote the rank of the root system (and the multiplicity of the related integral) as d , i.e. we consider the root system C_d .

Theorem 4. *Let nine parameters $t, t_a (a = 1, \dots, 6), p, q \in \mathbb{C}$ and seven discrete variables $n, n_a \in \mathbb{Z}$ satisfy the constraints $|p|, |q|, |t|, |t_a| < 1$, and the balancing condition*

$$t^{2d-2} \prod_{a=1}^6 t_a = pq, \quad 4n(d-1) + \sum_{a=1}^6 n_a = 0.$$

Then

$$\begin{aligned} \kappa_d^{(r)} \sum_{m_1, \dots, m_d=0}^{r-1} \int_{\mathbb{T}^d} \prod_{1 \leq j < k \leq d} \frac{\Gamma^{(r)}(t z_j^{\pm 1} z_k^{\pm 1}, 2n \pm m_j \pm m_k)}{\Gamma^{(r)}(z_j^{\pm 1} z_k^{\pm 1}, \pm m_j \pm m_k)} \prod_{j=1}^d \frac{\prod_{a=1}^6 \Gamma^{(r)}(t_a z_j^{\pm 1}, n_a \pm m_j)}{\Gamma^{(r)}(z_j^{\pm 2}, \pm 2m_j)} \frac{dz_j}{z_j} \\ = \prod_{j=1}^d \left(\frac{\Gamma^{(r)}(t^j, 2nj)}{\Gamma^{(r)}(t, 2n)} \prod_{1 \leq a < b \leq 6} \Gamma^{(r)}(t^{j-1} t_a t_b, 2n(j-1) + n_a + n_b) \right). \end{aligned} \quad (69)$$

Proof. We assume that the variables t_6 and n_6 are excluded with the help of the balancing condition and denote the integral on the left-hand side of (69) as $I_d^{(r)}(t, t_1, \dots, t_5, 2n, n_1, \dots, n_5)$.

Consider now the following $(2n - 1)$ -tuple integral

$$\begin{aligned}
& \kappa_d^{(r)} \kappa_{d-1}^{(r)} \sum_{m_1, \dots, m_d=0}^{r-1} \sum_{l_1, \dots, l_{d-1}=0}^{r-1} \int_{\mathbb{T}^{2d-1}} \prod_{j=1}^d \frac{dz_j}{z_j} \prod_{k=1}^{d-1} \frac{dw_k}{w_k} \\
& \times \prod_{1 \leq j < k \leq d} \frac{1}{\Gamma^{(r)}(z_j^{\pm 1} z_k^{\pm 1}, \pm m_j \pm m_k)} \prod_{j=1}^n \frac{\prod_{a=0}^5 \Gamma^{(r)}(t_a z_j^{\pm 1}, n_a \pm m_j)}{\Gamma^{(r)}(z_j^{\pm 2}, \pm 2m_j)} \\
& \times \prod_{\substack{1 \leq j \leq d \\ 1 \leq k \leq d-1}} \Gamma^{(r)}(t^{1/2} z_j^{\pm 1} w_k^{\pm 1}, n \pm m_j \pm l_k) \prod_{1 \leq j < k \leq d-1} \frac{1}{\Gamma^{(r)}(w_j^{\pm 1} w_k^{\pm 1}, \pm l_j \pm l_k)} \\
& \times \prod_{k=1}^{d-1} \frac{\Gamma^{(r)}(w_k^{\pm 1} t^{d-3/2} \prod_{a=1}^5 t_a, (2d-3)n + \sum_{a=1}^5 n_a \pm l_k)}{\Gamma^{(r)}(w_k^{\pm 2}, \pm 2l_k) \Gamma^{(r)}(w_k^{\pm 1} t^{2d-3/2} \prod_{a=1}^5 t_a, (4d-3)n + \sum_{a=1}^5 n_a \pm l_k)} \quad (70)
\end{aligned}$$

with $|t|, |t_a| < 1$ ($a = 0, \dots, 5$) and

$$t^{d-1} \prod_{a=0}^5 t_a = pq, \quad 2n(d-1) + \sum_{a=0}^5 n_a = 0.$$

Integration over the variables w_k with the help of formula (54) brings expression (70) to the form

$$\frac{\Gamma^{(r)}(t, 2n)^d}{\Gamma^{(r)}(t^d, 2nd)} I_n^{(r)}(t, t_1, \dots, t_5, 2n, n_1, \dots, n_5)$$

(where it is assumed that $t_6 = pqt^{2-2d} / \prod_{j=1}^5 t_j$). Because the integrand is bounded on the contour of integration, we can change the order of integrations. Then the integration over z_k -variables with the help of formula (54) converts expression (70) to

$$\Gamma^{(r)}(t; p, q)^{d-1} \prod_{0 \leq a < b \leq 5} \Gamma^{(r)}(t_a t_b, n_a + n_b) I_{d-1}^{(r)}(t, t^{1/2} t_1, \dots, t^{1/2} t_5, 2n, n + n_1, \dots, n + n_5).$$

As a result, we obtain a recurrence relation connecting integrals of different dimension d :

$$\frac{I_d^{(r)}(t, t_1, \dots, t_5, 2n, n_1, \dots, n_5)}{I_{d-1}^{(r)}(t, t^{1/2} t_1, \dots, t^{1/2} t_5, 2n, n + n_1, \dots, n + n_5)} = \frac{\Gamma^{(r)}(t^d, 2nd)}{\Gamma^{(r)}(t, 2n)} \prod_{0 \leq a < b \leq 5} \Gamma^{(r)}(t_a t_b, n_a + n_b).$$

Using known initial condition at $d = 1$ (51), we find (69) by recursion. \square

Evidently, for $d = 1$ both multiple integrals (54) and (69) reduce to the rarefied elliptic beta integral (51). We remark that relation (69), representing the most complicated known generalization of the Selberg integral, depends on the even integer variable $2n$ which is an artifact of the proposed proof.

Conjecture 1. *We conjecture that the exact integration formula (69) remains true after the replacement of the even integer $2n$ by an arbitrary integer $n \in \mathbb{Z}$.*

7. AN ANALOGUE OF THE EULER-GAUSS HYPERGEOMETRIC FUNCTION

Consider the double sum-integral

$$\sum_{m=0}^{r-1} \sum_{l=0}^{r-1} \int_{\mathbb{T}^2} \Gamma^{(r)}(fz^{\pm 1}w^{\pm 1}, h \pm m \pm l) \\ \times \frac{\prod_{a=1}^4 \Gamma^{(r)}(t_a z^{\pm 1}, n_a \pm m) \Gamma^{(r)}(s_a w^{\pm 1}, k_a \pm l)}{\Gamma^{(r)}(z^{\pm 2}, \pm 2m) \Gamma^{(r)}(w^{\pm 2}, \pm 2l)} \frac{dz}{z} \frac{dw}{w},$$

where $n_a, k_a, h \in \mathbb{Z}$, the variables $t_a, s_a, f \in \mathbb{C}^*$ satisfy the constraints $|t_a|, |s_a|, |f| < 1$ and the balancing conditions

$$f^2 \prod_{a=1}^4 t_a = f^2 \prod_{a=1}^4 s_a = pq, \quad 2h + \sum_{a=1}^4 n_a = 2h + \sum_{a=1}^4 k_a = 0. \quad (71)$$

Since h is an integer, we see that the balancing condition for discrete variables (71) forces the sums $\sum_{a=1}^4 n_a$ and $\sum_{a=1}^4 k_a$ to be even integers.

Because of the imposed constraints the contour \mathbb{T} is legitimate for computing sum-integrals over (z, m) or (w, ℓ) with the help of formula (51). Integrate first over z and sum over m . Then, using Fubini's theorem, we change the order and integrate first over w and sum over l . This yields the following identity.

$$\prod_{1 \leq a < b \leq 4} \Gamma^{(r)}(t_a t_b, n_a + n_b) \sum_{l=0}^{r-1} \int_{\mathbb{T}} \frac{\prod_{a=1}^4 \Gamma^{(r)}(f t_a w^{\pm 1}, h + n_a \pm l) \Gamma^{(r)}(s_a w^{\pm 1}, k_a \pm l)}{\Gamma^{(r)}(w^{\pm 2}, \pm 2l)} \frac{dw}{w} \quad (72) \\ = \prod_{1 \leq a < b \leq 4} \Gamma^{(r)}(s_a s_b, k_a + k_b) \sum_{m=0}^{r-1} \int_{\mathbb{T}} \frac{\prod_{a=1}^4 \Gamma^{(r)}(t_a z^{\pm 1}, n_a \pm m) \Gamma^{(r)}(f s_a z^{\pm 1}, h + k_a \pm m)}{\Gamma^{(r)}(z^{\pm 2}, \pm 2m)} \frac{dz}{z}.$$

Let us define the rarefied elliptic hypergeometric function

$$V^{(r)}(t_a, n_a | p, q) := \frac{(p^r; p^r)_{\infty} (q^r; q^r)_{\infty}}{4\pi i} \sum_{m=0}^{r-1} \int_{\mathbb{T}} \frac{\prod_{a=1}^8 \Gamma^{(r)}(t_a z^{\pm 1}, n_a \pm m)}{\Gamma^{(r)}(z^{\pm 2}, \pm 2m)} \frac{dz}{z}, \quad (73)$$

where $t_a \in \mathbb{C}^*$, $|t_a| < 1$, $n_a \in \mathbb{Z}$, and

$$\prod_{a=1}^8 t_a = (pq)^2, \quad \sum_{a=1}^8 n_a = 0. \quad (74)$$

As usual, the contour \mathbb{T} separates geometric progressions of poles converging to zero from their partners going to infinity. Other domains of values of the parameters are reached by analytical continuation. Both, the function itself (73) and the balancing condition (74), are invariant with respect to the group S_8 permuting parameters t_a and n_a (which is the Weyl group of the root system A_7). For $r = 1$ this is the elliptic analogue of the Euler-Gauss hypergeometric function introduced in [13], $V^{(1)}(t_a, n_a | p, q) \equiv V(t_a | p, q)$.

Suppose that parameters t_7, t_8, n_7 , and n_8 satisfy the constraints $t_7 t_8 = pq$ and $n_7 + n_8 = 0$. Then we have

$$\Gamma^{(r)}(t_7 z, n_7 + m) \Gamma^{(r)}(t_8 z^{-1}, n_8 - m) = 1$$

and these parameters drop out completely from the $V^{(r)}$ -function, which thus becomes equal to the rarefied elliptic beta integral.

Quasiperiodicity of the rarefied elliptic gamma function leads to the relation

$$\frac{V^{(r)}(\dots, n_b + r, \dots, n_c - r, \dots | p, q)}{V^{(r)}(t_a, n_a | p, q)} = [t_b^{r+2n_b} t_c^{r-2n_c} (p^{1-n_b-n_c} q^{1+n_b+n_c})^{n_c-n_b-r}]^{r-1}. \quad (75)$$

For odd r this relation allows one to convert all n_a to even numbers, i.e. without loss of generality, for odd r we can assume that all n_a are even.

Substituting definition (73) into relation (72), we obtain the transformation property of the V -function

$$V^{(r)}(t_a, n_a | p, q) = \prod_{1 \leq b < c \leq 4} \Gamma^{(r)}(t_b t_c, n_b + n_c) \Gamma^{(r)}(t_{b+4} t_{c+4}, n_{b+4} + n_{c+4}) V^{(r)}(s_a, k_a | p, q), \quad (76)$$

where

$$\begin{cases} s_a = \varepsilon^{-1} t_a, & k_a = n_a - \frac{1}{2} \sum_{b=1}^4 n_b, & a = 1, 2, 3, 4 \\ s_a = \varepsilon t_a, & k_a = n_a + \frac{1}{2} \sum_{b=1}^4 n_b, & a = 5, 6, 7, 8 \end{cases}; \quad \varepsilon = \sqrt{\frac{t_1 t_2 t_3 t_4}{pq}} = \sqrt{\frac{pq}{t_5 t_6 t_7 t_8}}$$

and $|t_a|, |s_a| < 1$. This is the key reflection transformation in the space of continuous parameters t_a extending the permutation group S_8 to the Weyl group of the exceptional root system E_7 . However, in the space of discrete variables n_a the situation is more complicated. Indeed, the above transformation is valid only when $\sum_{b=1}^4 n_b$ is an even integer — the condition which breaks S_8 to a smaller group of symmetries. S_8 -group is restored if we assume that the sum of arbitrary four integers n_a is even, which is possible if all n_a are either even or odd. As mentioned above, for odd r such a condition does not alter the generality of consideration. However, for even r , when there are both even and odd integers among n_a , we do not have full $W(E_7)$ group of symmetries. The detailed analysis of these particular number theoretical subgroups of $W(E_7)$ lies beyond the scope of this work.

Suppose all n_a are either even or odd. If $\frac{1}{2} \sum_{b=1}^4 n_b$ is an odd integer, then all discrete variables k_a are also even or odd with opposite oddness properties in comparison to n_a . Now the situation is completely similar to the standard V -function case. Therefore there are two more distinguished forms of the $W(E_7)$ -group transformations. The second transformation is obtained after repeating the first transformation with $t_{3,4,5,6}$ and $n_{3,4,5,6}$ playing the role of $t_{1,2,3,4}$ and $n_{1,2,3,4}$ and symmetrization of the resulting relation:

$$V^{(r)}(t_a, n_a | p, q) = \prod_{1 \leq b, c \leq 4} \Gamma^{(r)}(t_b t_{c+4}, n_b + n_{c+4}) V^{(r)}(s_a, k_a | p, q), \quad (77)$$

where

$$\begin{cases} s_a = t_a^{-1} \sqrt{t_1 t_2 t_3 t_4}, & k_a = \frac{1}{2} \sum_{\ell=1}^4 n_\ell - n_a, & a = 1, 2, 3, 4, \\ s_a = t_a^{-1} \sqrt{t_5 t_6 t_7 t_8}, & k_a = \frac{1}{2} \sum_{\ell=5}^8 n_\ell - n_a, & a = 5, 6, 7, 8, \end{cases}$$

and $|t_a|, |s_a| < 1$.

The third transformation is obtained after equating the right-hand side expressions in (76) and (77):

$$V^{(r)}(t_a, n_a | p, q) = \prod_{1 \leq b < c \leq 8} \Gamma^{(r)}(t_b t_c, n_b + n_c) V^{(r)}\left(\frac{\sqrt{pq}}{t_a}, -n_a | p, q\right). \quad (78)$$

For $r = 1$ all three relations become the standard symmetry transformations for the V -function with the key generating relation (76) derived in [13].

Let us shift $n_b \rightarrow n_b + r$ and $n_c \rightarrow n_c - r$ for arbitrary $b \neq c$ in the identity (78). One can check that the quasiperiodicity factors emerging from relation (75) cancel out. For odd r this proves that equality (78) is true for arbitrary integers n_a , not just when all of them are either odd or even.

Conjecture 2. *We conjecture that for even r the third symmetry transformation (78) is also true for arbitrary integers n_a (not only even or odd) satisfying the balancing condition.*

Let us derive the rarefied analogue of the elliptic hypergeometric equation constructed first in author's thesis [14] and published later in [16]. The (quartic) addition formula for elliptic theta functions can be written in the form

$$t_3 \theta(t_2 t_3^{\pm 1}, t_1 z^{\pm 1}; q^r) + t_1 \theta(t_3 t_1^{\pm 1}, t_2 z^{\pm 1}; q^r) + t_2 \theta(t_1 t_2^{\pm 1}, t_3 z^{\pm 1}; q^r) = 0. \quad (79)$$

It yields the following contiguous relation for the $V^{(r)}$ -function

$$\begin{aligned} & \frac{t_1^{1+2n_1} q^{-n_1(n_1+2)} V^{(r)}(pt_1, n_1 - 1)}{\theta(t_1 t_2^{\pm 1} q^{-n_1 \mp n_2}, t_1 t_3^{\pm 1} q^{-n_1 \mp n_3}; q^r)} + \frac{t_2^{1+2n_2} q^{-n_2(n_2+2)} V^{(r)}(pt_2, n_2 - 1)}{\theta(t_2 t_1^{\pm 1} q^{-n_2 \mp n_1}, t_2 t_3^{\pm 1} q^{-n_2 \mp n_3}; q^r)} \\ & + \frac{t_3^{1+2n_3} q^{-n_3(n_3+2)} V^{(r)}(pt_3, n_3 - 1)}{\theta(t_3 t_1^{\pm 1} q^{-n_3 \mp n_1}, t_3 t_2^{\pm 1} q^{-n_3 \mp n_2}; q^r)} = 0, \end{aligned} \quad (80)$$

where $V^{(r)}(pt_b, n_b - 1)$ denotes the $V^{(r)}(t_a, n_a)$ -function with the parameters t_b, n_b replaced by $pt_b, n_b - 1$ (with the balancing condition being $\prod_{a=1}^8 t_a = pq^2$, $\sum_{a=1}^8 n_a = 1$). Indeed, if we replace in (80) $V^{(r)}$ -functions by their integrands, then we obtain the equality

$$\begin{aligned} & \frac{t_1 q^{-n_1} \theta(t_1 z^{\pm 1} q^{-n_1 \mp m}; q^r)}{\theta(t_1 t_2^{\pm 1} q^{-n_1 \mp n_2}, t_1 t_3^{\pm 1} q^{-n_1 \mp n_3}; q^r)} + \frac{t_2 q^{-n_2} \theta(t_2 z^{\pm 1} q^{-n_2 \mp m}; q^r)}{\theta(t_2 t_1^{\pm 1} q^{-n_2 \mp n_1}, t_2 t_3^{\pm 1} q^{-n_2 \mp n_3}; q^r)} \\ & + \frac{t_3 q^{-n_3} \theta(t_3 z^{\pm 1} q^{-n_3 \mp m}; q^r)}{\theta(t_3 t_1^{\pm 1} q^{-n_3 \mp n_1}, t_3 t_2^{\pm 1} q^{-n_3 \mp n_2}; q^r)} = 0 \end{aligned}$$

multiplied by the function

$$z^{2m} q^{m^2} \prod_{a=1}^8 \frac{\Gamma^{(r)}(t_a z^{\pm 1}, n_a \pm m)}{\Gamma^{(r)}(z^{\pm 2}, \pm 2m)}.$$

Replacing $t_a \rightarrow t_a q^{n_a}$ and $z \rightarrow z q^m$ and simplifying the factors we obtain the addition formula (79). Integrating the resulting equation for the integrand functions over $z \in \mathbb{T}$ and summing in m , we come to (80). Note that for $r = 1$ one can pull out all powers of q out of the theta functions and find that all three terms in (80) get equal multipliers

$q^{\sum_{a=1}^3 n_a^2} / \prod_{a=1}^3 t_a^{2n_a}$, so that the dependence of this contiguous relation on n_a disappears completely.

Substituting relation (78) in (80), we obtain

$$\begin{aligned} & \frac{p^{2n_1} q^{n_1(n_1+3)}}{t_1^{2+2n_1}} \frac{\prod_{a=4}^8 \theta\left(\frac{t_1 t_a}{pq} q^{-n_1-n_a}; q^r\right)}{\theta\left(\frac{t_2}{t_1} q^{n_1-n_2}, \frac{t_3}{t_1} q^{n_1-n_3}; q^r\right)} V(p^{-1} t_1, n_1 + 1) \\ & + \frac{p^{2n_2} q^{n_2(n_2+3)}}{t_2^{2+2n_2}} \frac{\prod_{a=4}^8 \theta\left(\frac{t_2 t_a}{pq} q^{-n_2-n_a}; q^r\right)}{\theta\left(\frac{t_1}{t_2} q^{n_2-n_1}, \frac{t_3}{t_2} q^{n_2-n_3}; q^r\right)} V(p^{-1} t_2, n_2 + 1) \\ & + \frac{p^{2n_3} q^{n_3(n_3+3)}}{t_3^{2+2n_3}} \frac{\prod_{a=4}^8 \theta\left(\frac{t_3 t_a}{pq} q^{-n_3-n_a}; q^r\right)}{\theta\left(\frac{t_2}{t_3} q^{n_3-n_2}, \frac{t_1}{t_3} q^{n_3-n_1}; q^r\right)} V(p^{-1} t_3, n_3 + 1) = 0, \end{aligned} \quad (81)$$

where $\prod_{a=1}^8 t_a = p^3 q^2$ and $\sum_{a=1}^8 n_a = -1$. Shifting $t_3 \rightarrow pt_3$ and $n_3 \rightarrow n_3 - 1$ in (81), we obtain

$$\begin{aligned} & \frac{p^{2n_1} q^{n_1(n_1+3)}}{t_1^{2+2n_1}} \frac{\prod_{a=4}^8 \theta\left(\frac{t_1 t_a}{pq} q^{-n_1-n_a}; q^r\right)}{\theta\left(\frac{t_2}{t_1} q^{n_1-n_2}, \frac{pqt_3}{t_1} q^{n_1-n_3}; q^r\right)} V(p^{-1} t_1, pt_3, n_1 + 1, n_3 - 1) \\ & + \frac{p^{2n_2} q^{n_2(n_2+3)}}{t_2^{2+2n_2}} \frac{\prod_{a=4}^8 \theta\left(\frac{t_2 t_a}{pq} q^{-n_2-n_a}; q^r\right)}{\theta\left(\frac{t_1}{t_2} q^{n_2-n_1}, \frac{pqt_3}{t_2} q^{n_2-n_3}; q^r\right)} V(p^{-1} t_2, pt_3, n_2 + 1, n_3 - 1) \\ & + \frac{p^{2(n_3-1)} q^{(n_3-1)(n_3+2)}}{(pt_3)^{2n_3}} \frac{\prod_{a=4}^8 \theta(t_3 t_a q^{-n_3-n_a}; q^r)}{\theta\left(\frac{t_2}{pqt_3} q^{n_3-n_1}, \frac{t_1}{pqt_3} q^{n_3-n_1}; q^r\right)} V(t_a, n_a) = 0, \end{aligned}$$

Replacing $t_1 \rightarrow p^{-1} t_1$, $n_1 \rightarrow n_1 + 1$ or $t_2 \rightarrow p^{-1} t_2$, $n_2 \rightarrow n_2 + 1$ in (80) we obtain relations

$$\begin{aligned} & \frac{(p^{-1} t_1)^{3+2n_1} q^{-(n_1+1)(n_1+3)} V^{(r)}(t_a, n_a)}{\theta(p^{-1} t_1 t_2^{\pm 1} q^{-n_1-1 \mp n_2}, p^{-1} t_1 t_3^{\pm 1} q^{-n_1-1 \mp n_3}; q^r)} \\ & + \frac{t_2^{1+2n_2} q^{-n_2(n_2+2)} V^{(r)}(p^{-1} t_1, pt_2, n_1 + 1, n_2 - 1)}{\theta(t_2(p^{-1} t_1)^{\pm 1} q^{-n_2 \mp (n_1+1)}, t_2 t_3^{\pm 1} q^{-n_2 \mp n_3}; q^r)} \\ & + \frac{t_3^{1+2n_3} q^{-n_3(n_3+2)} V^{(r)}(p^{-1} t_1, pt_3, n_1 + 1, n_3 - 1)}{\theta(t_3(p^{-1} t_1)^{\pm 1} q^{-n_3 \mp (n_1+1)}, t_3 t_2^{\pm 1} q^{-n_3 \mp n_2}; q^r)} = 0, \end{aligned}$$

or

$$\begin{aligned} & \frac{t_1^{1+2n_1} q^{-n_1(n_1+2)} V^{(r)}(pt_1, p^{-1} t_2, n_1 - 1, n_2 + 1)}{\theta(t_1(p^{-1} t_2)^{\pm 1} q^{-n_1 \mp (n_2+1)}, t_1 t_3^{\pm 1} q^{-n_1 \mp n_3}; q^r)} \\ & + \frac{(p^{-1} t_2)^{3+2n_2} q^{-(n_2+1)(n_2+3)} V^{(r)}(t_a, n_a)}{\theta(p^{-1} t_2 t_1^{\pm 1} q^{-n_2-1 \mp n_1}, p^{-1} t_2 t_3^{\pm 1} q^{-n_2-1 \mp n_3}; q^r)} \\ & + \frac{t_3^{1+2n_3} q^{-n_3(n_3+2)} V^{(r)}(p^{-1} t_2, pt_3, n_2 + 1, n_3 - 1)}{\theta(t_3 t_1^{\pm 1} q^{-n_3 \mp n_1}, t_3(p^{-1} t_2)^{\pm 1} q^{-n_3 \mp (n_2+1)}; q^r)} = 0. \end{aligned}$$

Excluding now from the latter three equalities the functions $V^{(r)}(p^{-1}t_1, pt_3, n_1 + 1, n_3 - 1)$ and $V^{(r)}(p^{-1}t_2, pt_3, n_2 + 1, n_3 - 1)$, we come to the rarefied elliptic hypergeometric equation:

$$\begin{aligned} & \mathcal{A}\left(\frac{t_1}{q^{n_1}}, \frac{t_2}{q^{n_2}}, \dots, \frac{t_8}{q^{n_8}}, p; q^r\right) \left(U(pt_1, p^{-1}t_2, n_1 - 1, n_2 + 1) - U(t_a, n_a)\right) \\ & + \mathcal{A}\left(\frac{t_2}{q^{n_2}}, \frac{t_1}{q^{n_1}}, \dots, \frac{t_8}{q^{n_8}}, p; q^r\right) \left(U(p^{-1}t_1, pt_2, n_1 + 1, n_2 - 1) - U(t_a, n_a)\right) + U(t_a, n_a) = 0, \end{aligned} \quad (82)$$

where we have denoted

$$\mathcal{A}(t_1, \dots, t_8, p; q^r) := \frac{\theta\left(\frac{t_1}{pq^{1-r}t_3}, t_3t_1, \frac{t_3}{t_1}; q^r\right)}{\theta\left(\frac{t_1}{t_2}, \frac{t_2}{pq^{1-r}t_1}, \frac{t_1t_2}{pq^{1-r}}; q^r\right)} \prod_{a=4}^8 \frac{\theta\left(\frac{t_2t_a}{pq^{1-r}}; q^r\right)}{\theta(t_3t_a; q^r)} \quad (83)$$

and

$$U(t_a, n_a) := \frac{V^{(r)}(t_a, n_a)}{\prod_{k=1}^2 \Gamma^{(r)}(t_k t_3^{\pm 1}, n_k \pm n_3)}.$$

A fundamental fact is that for any r the function $\mathcal{A}(t_1, \dots, t_8, p; q^r)$ is a q^r -elliptic function of all parameters t_1, \dots, t_8 (one of which should be counted as a dependent variable through the balancing condition $\prod_{a=1}^8 t_a = (pq)^2$), i.e. it does not change after the scaling $t_a \rightarrow t_a q^r, t_b \rightarrow t_b q^{-r}$ for any $a \neq b$.

We call equation (82) the rarefied elliptic hypergeometric equation, though it does not have the form one would be willing to see. It can be checked that under the shifts $n_a \rightarrow n_a + r, n_b \rightarrow n_b - r, a \neq b$ the functions $U(p^{\pm 1}t_1, p^{\mp 1}t_2, n_1 \mp 1, n_2 \pm 1)$ and $U(t_a, n_a)$ have the same quasiperiodicity multipliers. Therefore, by shifting $t_{1,2} \rightarrow p^{\pm l}t_{1,2}, n_{1,2} \rightarrow n_{1,2} \mp l, l = 1, 2, \dots$, in equation (82), combining the resulting equation in an appropriate way and using the fact that

$$U(t_a, n_1 - r, n_2 + r, n_3, \dots) = U(t_a, n_1 + r, n_2 - r, n_3, \dots) = U(t_a, n_a),$$

one can derive the following tridiagonal equation

$$\alpha(t_a, n_a)U(p^r t_1, p^{-r} t_2, n_a) + \beta(t_a, n_a)U(t_a, n_a) + \gamma(t_a, n_a)U(p^{-r} t_1, p^r t_2, n_a) = 0, \quad (84)$$

for some coefficients α, β, γ . After parametrization $t_1 = cx, t_2 = cx^{-1}$ the latter equation becomes a “ q ”-difference equation of the second order for the variable x with “ q ” = p^r . It is appropriate to call equation (84) the rarefied elliptic hypergeometric equation, however, we could not derive a compact form of its coefficients yet. Note that we know already one of its solutions given by the $V^{(r)}$ -function. Its second linearly independent solution is obtained by application of the symmetry transformation of the equation which is not a symmetry of the solution, e.g. by multiplying its parameters by the powers q^r or some other means. The second order finite-difference equation (84) represents currently the most general equation of such type with the closed form solutions (an “exactly solvable” equation).

Actually, similar to the standard $r = 1$ case, equation (82) has a partner obtained by permuting the bases p and q :

$$\begin{aligned} & \mathcal{A}(t_1 p^{n_1}, t_2 p^{n_2}, \dots, t_8 p^{n_8}, q; p^r) \left(U(qt_1, q^{-1}t_2, n_1 + 1, n_2 - 1) - U(t_a, n_a) \right) + U(t_a, n_a) \\ & + \mathcal{A}(t_2 p^{n_2}, t_1 p^{n_1}, \dots, t_8 p^{n_8}, q; p^r) \left(U(q^{-1}t_1, qt_2, n_1 - 1, n_2 + 1) - U(t_a, n_a) \right) = 0. \end{aligned} \quad (85)$$

Let us denote

$$t_1 := cx, \quad t_2 := \frac{c}{x}, \quad \text{or} \quad c = \sqrt{t_1 t_2}, \quad x = \sqrt{\frac{t_1}{t_2}}$$

and

$$n_1 := n_c + n, \quad n_2 := n_c - n, \quad \text{or} \quad n_c = \frac{n_1 + n_2}{2}, \quad n = \frac{n_1 - n_2}{2}.$$

Now we introduce new continuous variables

$$\varepsilon_1 := \frac{c}{t_3 p q^{1-r}}, \quad \varepsilon_2 := \frac{c}{t_3}, \quad \varepsilon_3 := c t_3 q^{4r}, \quad \varepsilon_a := \frac{p q^{1-r}}{c t_a}, \quad a = 4, \dots, 8,$$

and new discrete variables

$$k_1 = k_2 := n_c - n_3, \quad k_3 := n_c + n_3, \quad k_a = -n_c - n_a, \quad a = 4, \dots, 8.$$

To keep integrality of the numbers k_a we have to assume that all n_a are either odd or even. It is easy to see that the balancing condition remains intact

$$\prod_{a=1}^8 \varepsilon_a = p^2 q^2, \quad \sum_{a=1}^8 k_a = 0.$$

Replacing $U(t_a, n_a)$ by unknown function $f(x, n)$, we obtain another form of the rarefied elliptic hypergeometric equation:

$$A(x q^{-n}) (f(px, n-1) - f(x, n)) + A(x^{-1} q^n) (f(p^{-1}x, n+1) - f(x, n)) + \nu f(x, n) = 0, \quad (86)$$

where

$$A(x) = \frac{\prod_{a=1}^8 \theta\left(\frac{\varepsilon_a x}{q^{k_a}}; q^r\right)}{\theta(x^2, p q^{1-r} x^2; q^r)}, \quad \nu = \prod_{a=3}^8 \theta\left(\frac{\varepsilon_1 \varepsilon_a}{q^{k_1 + k_a}}; q^r\right). \quad (87)$$

Define now a C_d -root system analogue of the rarefied elliptic hypergeometric function (73). Take two bases $p, q \in \mathbb{C}$, $|q|, |q| < 1$, and 18 continuous and discrete parameters $t, t_a \in \mathbb{C}^*$ and $n, n_a \in \mathbb{Z}$ ($a = 1, \dots, 8$) and impose the balancing condition

$$t^{2d-2} \prod_{a=1}^8 t_a = (pq)^2, \quad (2d-2)n + \sum_{a=1}^8 n_a = 0. \quad (88)$$

The C_d -extension of the V -function is

$$V^{(r)}(t, t_a, n, n_a | p, q) := \frac{(p^r; p^r)_\infty^d (q^r; q^r)_\infty^d}{d! (4\pi i)^d} \sum_{m_1, \dots, m_r=0}^{r-1} \int_{\mathbb{T}^d} \prod_{j=1}^d \frac{dz_j}{z_j} \quad (89)$$

$$\times \prod_{1 \leq j < k \leq d} \frac{\Gamma^{(r)}(t z_j^{\pm 1} z_k^{\pm 1}, n \pm m_j \pm m_k)}{\Gamma^{(r)}(z_j^{\pm 1} z_k^{\pm 1}, \pm m_j \pm m_k)} \prod_{j=1}^d \frac{\prod_{a=1}^8 \Gamma^{(r)}(t_a z_j^{\pm 1}, n_a \pm m_j)}{\Gamma^{(r)}(z_j^{\pm 2}, \pm 2m_j)},$$

where $|t|, |t_a| < 1$. For $r = 1$ this is the function introduced by Rains in [9].

Conjecture 3. *Suppose that all n_a are either even or odd integers. Then we conjecture that $V^{(r)}(t, t_a, n, n_a | p, q)$ obeys the $W(E_7)$ group of transformation symmetries generated by the following identity*

$$V^{(r)}(t, t_a, n, n_a | p, q) = \prod_{1 \leq a < b \leq 4} \prod_{l=0}^{d-1} \Gamma^{(r)}(t^l t_a t_b, l n + n_a + n_b) \\ \times \Gamma^{(r)}(t^l t_{a+4} t_{b+4}, l n + n_{a+4} + n_{b+4}) V^{(r)}(t, s_a, k_a | p, q), \quad (90)$$

where

$$\begin{cases} s_a = \varepsilon^{-1} t_a, & k_a = n_a - \rho, & a = 1, 2, 3, 4 \\ s_a = \varepsilon t_a, & k_a = n_a + \rho, & a = 5, 6, 7, 8 \end{cases}; \quad \varepsilon = \sqrt{\frac{t_1 t_2 t_3 t_4}{p q t^{1-d}}}, \quad \rho = \frac{1}{2} \sum_{b=1}^4 n_b + (d-1)n,$$

and $|t|, |t_a|, |s_a| < 1$ together with the condition that $\sum_{b=1}^4 n_b$ is even.

For $r = 1$ this is the Rains identity [9] and for $d = 1$ this is the relation (76) proven above.

Similar to the situation investigated in [14, 16], we consider the space of sequences of holomorphic functions of $z_j \in \mathbb{C}^*$, which are r -periodic in the discrete variables, $\varphi(z_j, \dots, m_k, m_k + r, m_{k+1}, \dots) = \varphi(z_j, m_j)$, and define the inner product for it

$$\langle \varphi, \psi \rangle = \kappa_d^{(r)} \sum_{m_1, \dots, m_d=0}^{r-1} \int_{\mathbb{T}^d} \Delta(z_k, m_k; t, t_a, n, n_a) \varphi(z_j, m_j) \psi(z_j, m_j) \prod_{k=1}^d \frac{dz_k}{z_k}$$

with the weight function

$$\Delta(z_k, m_k; t, t_a, n, n_a) = \prod_{1 \leq j < l \leq d} \frac{\Gamma(t z_j^{\pm 1} z_l^{\pm 1}, n \pm m_j \pm m_l)}{\Gamma(z_j^{\pm 1} z_l^{\pm 1}, \pm m_j \pm m_l)} \prod_{j=1}^d \frac{\prod_{a=1}^8 \Gamma(t_a z_j^{\pm 1}, n_a \pm m_j)}{\Gamma(z_j^{\pm 2}, \pm 2m_j)},$$

where $|t|, |t_a| < 1$. Let us introduce the finite-difference operator

$$\mathcal{D} = \sum_{j=1}^d \left(A_j(z_k q^{-m_k}) (T_{p,j} S_j^{-1} - 1) + A_j(z_k^{-1} q^{m_k}) (T_{p,j}^{-1} S_j - 1) \right),$$

$$A_j(z_k) = \frac{\prod_{a=1}^8 \theta(t_a q^{-n_a} z_j; q^r)}{\theta(z_j^2, p q^{1-r} z_j^2; q^r)} \prod_{l=1, \neq j}^d \frac{\theta(t q^{-n} z_j z_l^{\pm 1}; q^r)}{\theta(z_j z_l^{\pm 1}; q^r)}. \quad (91)$$

where $T_{p,j}^n S_j^m f(z_k, m_k) = f(\dots, p^n z_j, \dots, m_j + m, \dots)$ and assume validity of the balancing constraint (88). For $r = 1$ this is the Hamiltonian of the van Diejen completely integrable model [5] under the additional balancing condition. Suppose that the parameters t_a are constrained in such a way that the unit circle \mathbb{T} separates the sequences of poles converging to zero $z_j = 0$ in the expression $\langle \varphi, \mathcal{D}\psi \rangle$ from their partners going to infinity. Then the operator (91) is symmetric with respect to the above inner product,

$$\langle \varphi, \mathcal{D}\psi \rangle = \langle \mathcal{D}\varphi, \psi \rangle.$$

Surprisingly, this statement requires a rather complicated computation associated with the presence of the powers q^r in the arguments of theta functions. Because the suggested generalization of the van Diejen operator does not touch its analytical structure (or, more precisely, does not change the divisor structure of the functional coefficients entering it), our operator should define a completely integrable quantum many body system as well (in the sense that there exist d finite-difference operators of a similar form of the higher order in the shifting operators $T_{p,j} S_j^{-1}$). One can remove the balancing condition and consider a more general model, but this leads to a substantial complication of the form of the operator \mathcal{D} which requires a separate consideration.

Evidently, the function $f(z_k, m_k) = 1$ is a $\lambda = 0$ solution of the standard eigenvalue problem for the operator (91), $\mathcal{D}f(z_k, m_k) = \lambda f(z_k, m_k)$. The norm of this eigenfunction

$$\|1\|^2 = V^{(r)}(t, t_a, n, n_a | p, q) = \kappa_d^{(r)} \sum_{m_1, \dots, m_d=0}^{r-1} \int_{\mathbb{T}^d} \Delta(z_k, m_k; t, t_a, n, n_a) \prod_{j=1}^d \frac{dz_j}{z_j}$$

is exactly the multivariable analogue of the rarefied elliptic hypergeometric function for the root system C_d described above.

Another application to eigenvalue problems comes from comparing the operator \mathcal{D} with the rarefied elliptic hypergeometric equation in the form (86). One can see that the latter equation represents the eigenvalue problem for the operator \mathcal{D} with the following three special restrictions: 1) $d = 1$, 2) $t_2 = t_1 p q^{1-r}$, 3) $\lambda = -\nu$, i.e. the $d = 1$ V -function is now interpreted as a special eigenfunction of \mathcal{D} with a particular additional restriction on the parameters and valid for a special eigenvalue. This is completely similar to the situation taking place in the $r = 1$ case [14, 16].

8. CONCLUSION

In this paper we have proved several identities for the rarefied elliptic hypergeometric integrals and formulated a few related conjectures. Summarizing them it is natural to conjecture that all exact relations either proven [9, 17] or conjectured [19, 20] have rarefied analogues obtained simply by replacing the elliptic gamma functions $\Gamma(z; p, q)$ to $\Gamma^{(r)}(z, m; p, q)$ (with some mild restrictions on the parity of $m \in \mathbb{Z}$) and integrations $\int_{\mathbb{T}^d}$ to $\sum_{m_1, \dots, m_d=0}^{r-1} \int_{\mathbb{T}^d}$. If true, this yields a tremendous amount of new handbook formulae. Furthermore, they can be multiplied by considering various degeneration limits. Indeed, the elliptic hypergeometric integrals can be reduced to hyperbolic integrals [17], which corresponds to the reduction of $4d$ superconformal indices to $3d$ partition functions [10].

Applying a similar limit to the rarefied versions of these integrals, one gets the rarefied hyperbolic integrals, or $3d$ partition functions on the squashed lens space, which was mentioned already in [3]. In a similar fashion one can degenerate sums of integrals to terminating rarefied elliptic hypergeometric series.

Let us shift the discrete summation variables $m_\ell \rightarrow m_\ell - [r/2]$, where $[x]$ is the integer part of a real variable x , and take the limit $r \rightarrow \infty$. Such a limit describes a degeneration of superconformal indices on the lens space to $3d$ superconformal indices [3]. Again, this would yield a very large number of exact identities for corresponding infinite bilateral sums of q -hypergeometric integrals. As shown in [2], partition functions of $4d$ supersymmetric field theories on $S^1 \times S^3$ are equal to the corresponding superconformal indices up to an exponential of the Casimir energy. It is natural to expect that similar situation holds for partition functions on lens spaces and the rarefied elliptic hypergeometric integrals.

The integrals considered in this work are related only to the simplest lens space. It is possible to extend them to the general lens space, which adds more discrete parameters. As to other applications of our results, let us mention that it is not difficult to formulate a Bailey lemma based on the derived rarefied elliptic beta integral leading to the star-triangle relation and adapt the $2d$ integrable lattice model of [8] to present situation. Similarly, it is possible to derive a solution of the Yang-Baxter equation extending the R -operator of [4] to $r > 1$. The same technique can be used to generalize an elliptic hypergeometric integral identity used as the associativity condition in $2d$ topological field theories [10], etc. All these problems are currently in different stages of consideration.

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